On the Integrability of an Attracting System of Nonlinear Oscillators

H. Tasso

Max-Planck-Institut für Plasmaphysik,
Euratom Association, Garching bei München,
Federal Republic of Germany

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The problem of the integrability of a peculiar system of nonlinear oscillators is considered. While the case of two oscillators is integrable, the case of many is open. The Lax pair method is not applicable to such a system.

A system of nonlinear oscillators of an extended Van der Pol type was recently introduced [1] by the author. Under certain conditions the solutions of this system are attracted by the solutions of a nonlinear conservative system [1, 2]. It has been proved in [2] that the latter system is integrable in the case of two oscillators. For three or more oscillators the question of integrability is open. The purpose of this note is to investigate the difficulties that may be incurred in trying to answer this question.

The attracting system is of the form

$$\ddot{Y} + [\alpha(Y, Y)\, M_a + \beta(\dot{Y}, \dot{Y})\, N_a - P_a]\, \dot{Y} + \varepsilon Y = 0, \quad (1)$$

where Y is a real vector of arbitrary length r, α, β and ε are positive constants, and M_a, N_a and P_a are antisymmetric $r \times r$ matrices. For $r = 2$, system (1) can be integrated completely (see [2]) via complex plane representation and with the help of elliptic integrals. For $r > 2$, the complex plane method fails and direct integrability becomes hopeless, as can be seen for $r = 3$ in the following.

In this case the simplest choices of M_a, N_a and P_a are

$$M_a = \begin{pmatrix} 0 & \frac{m}{\alpha} & 0 \\ -\frac{m}{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N_a = \begin{pmatrix} 0 & \frac{n}{\beta} & 0 \\ -\frac{n}{\beta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_a = \begin{pmatrix} 0 & -p & 0 \\ p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

With choice (2), system (1) can be explicitly written in terms of the components of Y, i.e. y_1, y_2, y_3, as

$$\ddot{y}_1 + [m(y_1^2 + y_2^2 + y_3^2) + n(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2) + p]\, y_2 + \varepsilon y_1 = 0, \quad (3)$$

$$\ddot{y}_2 - [m(y_1^2 + y_2^2 + y_3^2) + n(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2) + p]\, y_1 + \varepsilon y_2 = 0, \quad (4)$$

$$\ddot{y}_3 + \varepsilon y_3 = 0. \quad (5)$$

If $y_3 = 0$, system (3), (4), (5) becomes equivalent to that considered in [2], which was proved to be integrable. But if $y_3 \neq 0$, only (5) is integrable. Equations (3) and (4) can be considered as non-autonomous in general because of the presence of $(m\, y_3^2 + n\, \dot{y}_3^2)$, which is time-dependent in a trigonometric way. In other words, (3) and (4) are equivalent to system (5) of [2] up to a time-dependent p. This time dependence would prevent the integration of (14) of [2].

Another way to prove integrability for large systems is the Lax pair method [3]. Let L be a symmetric operator (here a matrix) depending on $X = \begin{pmatrix} Y \\ \dot{Y} \end{pmatrix}$ in a linear way, and suppose that an anti-symmetric operator exists such that

$$\dot{L} = [A, L] \quad (6)$$

is equivalent to system (1). The eigenvalues of L are then integrals of motion. This method has been successfully applied to one-dimensional many-body canonical Hamiltonian systems in [4] and [5].

Unfortunately, this method has an inherent limitation which prevents its application to our case. This can be seen by taking the trace of (6):

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Trace}\,(L)) = 0. \quad (7)$$

From (7) it follows that

$$\sum \ddot{y}_i = 0, \quad (8)$$

if L is defined similarly to the case of [4]. But we know that (8) or any other linear equation is violated [2] by system (1) even for $r = 2$.

Reprint requests to Dr. H. Tasso, Max-Planck-Institut für Plasmaphysik, D-8046 Garching bei München.

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In fact, (8) is satisfied in the case of [4] because the system possesses a translation invariant Lagrangian (conservation of total momentum). Unfortunately, system (1) does not even possess a Lagrangian in terms of Y owing to the nonlinear terms in front of $\ddot{Y}$. System (1) is, however, basically conservative, i.e. an energy integral exists if a scalar product by $\dot{Y}$ is taken. In fact, the $(\dot{Y}, \ldots, \dot{Y})$ term disappears because of the antisymmetry of M_a, N_a and P_a. The $\dot{Y}$ term in system (1) acts as a nonlinear gyroscopic force but, unfortunately, is expressed in a non-Lagrangian way.

It is very likely that system (1) possesses a Lagrangian in some well-chosen variable Z and the whole difficulty is to find it explicitly. The Lax pair method would fail from the outset unless system (1) is written explicitly in Z. Even then it cannot be taken for granted that the Lagrangian is translation invariant and that a Lax pair exists such that (6) is satisfied.

Though these remarks do not yield a proof of nonintegrability, they indicate that it is unlikely that system (1) is integrable in general. This situation is somewhat reminiscent of celestial mechanics if one oscillator is considered as one body. There the two-body problem is integrable and the many-body problem is open.

The peculiarity of system (1) is that it is the attractor of a driven damped system previously introduced [1] by the author. The latter system describes a dissipative process which must end asymptotically in system (1). Such transient behaviour could have some importance, particularly in statistical thermodynamics, i.e. in the description of thermalization after contact with a heat bath. Other obvious applications could occur in circuits with active elements or in the kinetics of chemical reactions, etc.

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